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Removing One Element from an Exact Additive Basis

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We are interested in exact asymptotic bases. Our problem is to obtain an estimate from above for the exact order (which is known to exist) of such a basis

of Grekos.

Key Words: additive basis.

1. INTRODUCTION

This paper deals with the so-called exact (additive) asymptotic bases, that is, sets of integers $\mathscr{A}$ such that there exists an integer h satisfying $h\mathscr{A} \sim \mathbb{N}$. The exact order of $\mathscr{A}$, $\operatorname{ord}^*(\mathscr{A})$, is defined as the least such integer. As usual, we define

$$h\mathscr{A} = \{a_1 + \cdots + a_h \mid a_1, ..., a_h \in \mathscr{A}\}$$

and $\mathscr{A} \sim \mathscr{B}$ means that the symmetric difference of $\mathscr{A}$ and $\mathscr{B}$ is finite. The reader is referred to [6] for the general theory of additive bases and to [2] as a source of problems around this theory.

It is known [1, 4] that if $h\mathscr{A} \sim \mathbb{N}$, then for all but a finite number of elements a in $\mathscr{A}$, $\mathscr{A} \backslash \{a\}$ also is an exact asymptotic basis. We denote by $\mathscr{A}^*$ the set of elements of $\mathscr{A}$ having this property. An interesting quantity is

$$X(h) = \max_{h\mathscr{A} \sim \mathbb{N}} \ \max_{a \in \mathscr{A}^*} \operatorname{ord}^*(\mathscr{A} \backslash \{a\}).$$

[1] I am grateful to Georges Grekos who, in addition to drawing my attention to this problem, provided a careful reading of the paper.

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It has been shown [1, 9] that $X(2)=4$ and $X(3)=7$. The following example shows that $X(4)\geqslant 10$. If

$$\mathscr{B}=\{0\}\cup\{n \text{ such that } n\equiv 2 \text{ or } 5 \ (\mathrm{mod}\ 11)\},$$

then $4\mathscr{B}\sim\mathbb{N}$ and $k(\mathscr{B}\backslash\{0\})\nsim\mathbb{N}$ for any $k\leqslant 9$. Seemingly, Ref. [8] should be quoted for $X(4)\geqslant 10$ but we have not been able to obtain it. In general, only lower and upper bounds are known, namely

$$\frac{h^2-3h}{3}\leqslant X(h)\leqslant\frac{h^2+3h}{2}, \tag{1}$$

so that even the order of magnitude of X is unknown. In (1), the lower bound is due to Grekos [4] and the upper one to Nash [9] who employed the powerful Kneser's theorem. We also quote the following lower bound [10], which although less accurate than Grekos' one is nevertheless interesting for the small values of h,

$$\left\lceil\frac{h^2+6h+1}{4}\right\rceil\leqslant X(h). \tag{2}$$

In view of the constructions used to prove the lower bounds, we may think that only very few elements, when removed, leads to exact asymptotic bases with order $X(h)$. That is why Grekos introduced the following related quantity

$$S(h)=\max_{h\mathscr{A}\sim\mathbb{N}}\ \limsup_{a\in\mathscr{A}^*}\mathrm{ord}^*(\mathscr{A}\backslash\{a\}),$$

which is expected to be much smaller than $X(h)$ except for $h=1$ where trivially $S(1)=X(1)=1$. Unfortunately, in the general case, the best that is known is ($h\geqslant 2$),

$$h+1\leqslant S(h)\leqslant X(h),$$

the lower bound being due to Härtter [7]. Grekos' conjecture is that, for $h>1$,

$$S(h)<X(h),$$

a conjecture supported by $S(2)=3<X(2)$ and $S(3)\leqslant 6<X(3)$ (proved in [3, 5], respectively).

In this paper, we settle Grekos' conjecture for h large enough (namely for $h\geqslant 61$ in view of the lower bound from (1)) by proving the following more precise result.

THEOREM 1. *For any positive integer h, one has*

$$S(h) \leqslant \frac{h^2}{4} + 4h + 2.$$

2. SOME LEMMAS

We consider an exact (additive) asymptotic basis $\mathscr{A}$ with exact order $h \geqslant 2$,

$$\mathscr{A} = \{a_0 < a_1 < \cdots < a_k < \cdots\}.$$

Let $a \in \mathscr{A}$ be large enough (namely such that every integer larger than or equal to $a - a_1$ belongs to $h\mathscr{A}$) and consider $\mathscr{B} = \mathscr{A} \setminus \{a\}$. First, we notice that there is no loss of generality when assuming that $a_0 = 0$ since this corresponds to a translation of $\mathscr{A}$.

Our study is based on the function γ defined on $\{0, ..., h-1\}$ by $\gamma(0) = 0$, and, for $1 \leqslant i \leqslant h-1$,

$$\gamma(i) = \min\{k \in \mathbb{N}, \text{ such that there exist } b_1, ..., b_k \in \mathscr{B} \text{ with } ia = b_1 + \cdots + b_k\}.$$

The reason is that any integer n, larger than $ha + 1$, can be written in the form

$$n = sa + \alpha_1 + \cdots + \alpha_{h-s},$$

with $s < h$ and the α_j's different from a. This gives

LEMMA 1. $S(h) \leqslant \max_{0 \leqslant s \leqslant h-1}(\gamma(s) + h - s)$.

We now begin our study of γ with three crucial lemmas that give some indications on the behaviour of this function.

LEMMA 2. *Let i be an integer with $i \geqslant 1$.*

(i) *There exists an integer k with $0 \leqslant k \leqslant i-1$ such that*

$$\gamma(i) \leqslant h + 1 + \gamma(k) - k;$$

(ii) *there exists an integer j with $1 \leqslant j \leqslant i$ such that*

$$\gamma(j) \leqslant h + 1 + j - i.$$

Proof. The number $ia-a_1$ belongs to $h\mathscr{A}$ so that we can write it

$$ia-a_1=\alpha_1+\cdots+\alpha_h,$$

with the α_i's in $\mathscr{A}$. Let k be the number of α_i's that are equal to a. It is clear that $0\leqslant k\leqslant i-1$. Without loss of generality, we may assume that $\alpha_1=\cdots=\alpha_k=a$ which gives

$$ia=a_1+\alpha_{k+1}+\cdots+\alpha_h+ka,$$

and proves (i). Also

$$(i-k)\,a=a_1+\alpha_{k+1}+\cdots+\alpha_h\in(h-k+1)\,\mathscr{B}.$$

By writing $j=i-k$, we get (ii). ∎

LEMMA 3. *Let i, j, q, and r be integers satisfying $i=jq+r$. Then*

$$\gamma(i)\leqslant q\gamma(j)+\gamma(r).$$

Proof. This is trivial since $ia=q(ja)+(ra)$. ∎

LEMMA 4. *For $0\leqslant i\leqslant h-1$,*

$$\gamma(i)\leqslant(1+h)\,i-\frac{i(i-1)}{2}.$$

Proof. Since $\gamma(0)=0$, we assume $i\geqslant 1$. By Lemma 2(i), there exists an integer i_1 with $i_1<i$ such that

$$\gamma(i)\leqslant h+1+\gamma(i_1)-i_1.$$

Repeating this process, we find a strictly decreasing finite sequence of integers $(i_k)_{1\leqslant k\leqslant l}$ such that

$$\gamma(i_k)\leqslant h+1+\gamma(i_{k+1})-i_{k+1}$$

and $i_l=0$. Summing all these inequalities, we obtain

$$\gamma(i)\leqslant\sum_{k=1}^{l}(h+1-i_k).$$

The maximum of the right-hand side is obtained for $i_k=i-k$ and $l=i$. This gives the result. ∎

We immediately notice that this lemma together with Lemma 1 already provides

$$S(h) \leqslant \frac{h^2+3h}{2} - 1,$$

an estimate better by one than that of (1), obtained without the use of Kneser's theorem.

We now improve on Lemma 4 for the small values of i.

LEMMA 5. *If* $h \geqslant 1$ *and* $1 \leqslant i \leqslant \min(4(h+1)/7, h-1)$ *then*

$$\gamma(i) \leqslant h+1+(h+2)\, i - i^2.$$

Proof. We first notice that if $1 \leqslant i \leqslant 3$, the conclusion follows directly from Lemma 4. We therefore may assume $h \geqslant 5$ and $i \geqslant 4$. Note that this implies $\min(4(h+1)/7, h-1) = 4(h+1)/7$.

Let j be the integer $1 \leqslant j \leqslant i$ given by Lemma 2(ii) and perform the Euclidean division $i = jq + r$, with $q = [i/j]$ and $0 \leqslant r \leqslant j-1$.

If $q = 1$, then by Lemma 3

$$\begin{aligned} \gamma(i) &\leqslant \gamma(j) + \gamma(i-j) \\ &\leqslant h+1+j-i+(1+h)(i-j) - \frac{(i-j)(i-j-1)}{2}, \end{aligned}$$

by definition of j and Lemma 4. The function of $i-j$ appearing on the right-hand side is increasing on $\{0, ..., h\}$. Since $q=1$, one has $j > i/2$. Thus $i - j < i/2 (\leqslant h)$ and we obtain

$$\gamma(i) \leqslant h+1+\frac{hi}{2} - \frac{i(i-2)}{8}. \tag{3}$$

Consider now the case where $q \geqslant 2$ (or equivalently $j \leqslant i/2$). We have by Lemma 3, Lemma 4, the fact that the upper bound given by this lemma is increasing on the range involved and, finally, $r \leqslant j$,

$$\begin{aligned} \gamma(i) &\leqslant [i/j]\, \gamma(j) + \gamma(r) \\ &\leqslant \frac{i(h+1+j-i)}{j} + (1+h)\, r - \frac{r(r-1)}{2} \\ &\leqslant \frac{i(h+1+j-i)}{j} + (1+h)\, j - \frac{j(j-1)}{2} = \phi(j), \end{aligned}$$

say.

We want to show that ϕ is unimodal. Let us given the main lines of the proof of this. One has (since $1 \leqslant i \leqslant h-1$)

$$\phi'''(j) = -6\,\frac{i(h+1-i)}{j^4} < 0.$$

Now

$$\phi''(j) = 2\,\frac{i(h+1-i)}{j^3} - 1;$$

therefore $\phi''(1) = 2i(h+1-i) - 1 > 0$ and

$$\phi''(h-1) = 2\,\frac{i(h+1-i)}{(h-1)^3} - 1 \leqslant \frac{(h+1)^2}{2(h-1)^3} - 1 < 0.$$

This follows from $1 \leqslant i \leqslant h-1$ and $h \geqslant 4$, respectively. Thus ϕ' is first increasing and then decreasing on $\{1, 2, ..., h-1\}$. But $(2 \leqslant i \leqslant h-1,\ h \geqslant 3)$

$$\phi'(1) = h + \tfrac{1}{2} - i(h+1-i) \leqslant h + \tfrac{1}{2} - 2(h-1) < 0$$

and

$$\phi'(i/2) = \frac{11}{2} + h - \left(\frac{i}{2} + \frac{4(h+1)}{i}\right) > 0$$

for $4 \leqslant i \leqslant h-1$ and $h \geqslant 4$. This finally proves that ϕ also is unimodal, first decreasing and then increasing.

Thus

$$\gamma(i) \leqslant \max(\phi(1), \phi(i/2)). \tag{4}$$

Now, a straightforward computation shows that

$$\phi(1) \geqslant \phi(i/2) \tag{6}$$

is equivalent to $2 \leqslant i \leqslant 4(h+1)/7$ (for $h \geqslant 3$).

By (5), (3), (4), and the fact that

$$\phi(i/2) \geqslant h + 1 + \frac{hi}{2} - \frac{i(i-2)}{8},$$

we get

$$\gamma(i) \leqslant \phi(1) = i(h+2-i) + (1+h),$$

hence the result. ∎

3. PROOF OF THEOREM 1

If $h \leqslant 4$, the veracity of the theorem follows trivially from the Introduction. Consequently we may now assume $h \geqslant 5$, in which case

$$\frac{4(h+1)}{7} \leqslant h-1.$$

In view of Lemma 1, we have

$$S(h) \leqslant h-1+\max_{0 \leqslant s \leqslant h-1} (\gamma(s)).$$

We therefore focus our attention on $\max_{0 \leqslant s \leqslant h-1}(\gamma(s))$. We distinguish several cases.

First Case. $1 \leqslant s \leqslant 4(h+1)/7$.

The upper bound coming from Lemma 5 has its maximum in $h/2+1$. We thus get

$$\gamma(s) \leqslant \frac{h^2}{4}+2h+2.$$

Second Case. $4(h+1)/7 < s \leqslant h-1$.

In this case, write as before $s=jq+r$ with $1 \leqslant j \leqslant s$ given by Lemma 2(ii), $q=[s/j]$, and $0 \leqslant r \leqslant j-1$. We distinguish some subcases.

If $q=1$ then by Lemma 3, we have

$$\gamma(s) \leqslant h+1-r+\gamma(r).$$

But $r < s/2 \leqslant h/2 \leqslant 4(h+1)/7$ so we have by Lemma 5,

$$\begin{aligned} \gamma(s) &\leqslant h+1-r+h+1+(h+2)\,r-r^2 \\ &\leqslant \frac{(h+1)^2}{4}+2h+2, \end{aligned}$$

the value obtained when replacing r with $(h+1)/2$ (where the maximum of the function involved takes place).

If $q \geqslant 2$ then $r < j \leqslant s/2$. In this case, the upper bound given by Lemma 5 is an increasing function of its argument (for $r \leqslant h/2$). Applying Lemma 3 and Lemma 5, we thus have

$$\begin{aligned} \gamma(s) &\leqslant \frac{s(h+1+j-s)}{j}+\gamma(r) \\ &\leqslant \frac{s(h+1+j-s)}{j}+h+1+(h+2)\,j-j^2=\psi(j), \end{aligned}$$

say. In the same way as in the proof of Lemma 5, one can observe that $\psi'''(j)<0$, $\psi''(1)>0$ and $\psi''(h-1)<0$ proving thus that ψ' is first increasing and then decreasing. Now $\psi'(1)<0$ and $\psi'(s/2)>0$, this last fact being due to $s\geqslant\lceil 4(h+1)/7\rceil\geqslant 4$. Finally, we deduce from this that the function ψ is first decreasing and then increasing from where we derive that either

$$\gamma(s)\leqslant s(h+2-s)+2h+2\leqslant\frac{(h+2)^2}{4}+2h+2,$$

corresponding to $j=1$ or $(j=s/2)$

$$\begin{aligned}\gamma(s)&\leqslant 2(h+1-s/2)+h+1+\frac{(h+2)\,s}{2}-\frac{s^2}{4}\\&=3(h+1)+\frac{hs}{2}-\frac{s^2}{4}\\&\leqslant\frac{h^2}{4}+3h+2,\end{aligned}$$

for any $s\leqslant h$. This finishes the proof of the last case. Hence the theorem. ∎

Remark on the Method. Our method was to derive from the definition some properties of γ and then to forget from where this function comes and just solve an optimization problem. It is worth mentioning that the following system

$$\gamma(1)=\left[\frac{h}{2}\right],\qquad \gamma(2)=2\left[\frac{h}{2}\right],\ ...,$$
$$\gamma\left(\left[\frac{h}{2}\right]\right)=\left[\frac{h}{2}\right]^2,\qquad \gamma\left(\left[\frac{h}{2}\right]+1\right)=\cdots=\gamma(h-1)=1$$

verifies Lemmas 2 and 3 and would lead to $S(h)\sim h^2/4$. I don't state that such a system is possible, but proving that it is not possible will not follow from our method based on the lemmas. Thus we cannot expect anything better (except for the linear term that we did not try to optimize here) than Theorem 1 with this method.

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